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σ-derivations on generalized matrix algebras

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Abstract

Let $\mathcal{R}$ be a commutative ring with unity, $\mathcal{A}, \mathcal{B}$ be $\mathcal{R}$-algebras, $\mathcal{M}$ be $(\mathcal{A}, \mathcal{B})$-bimodule and $\mathcal{N}$ be $(\mathcal{B}, \mathcal{A})$-bimodule. The $\mathcal{R}$-algebra $\mathcal{G} = \mathcal{G}(\mathcal{A}, \mathcal{M}, \mathcal{N}, \mathcal{B})$ is a generalized matrix algebra defined by the Morita context $(\mathcal{A}, \mathcal{B}, \mathcal{M}, \mathcal{N}, \xi_{\mathcal{MN}}, \Omega_{\mathcal{NM}})$. In this article, we study Jordan σ-derivations on generalized matrix algebras.

1 Introduction

Let $\mathcal{R}$ be a commutative ring with unity and $\mathcal{A}$ be an $\mathcal{R}$-algebra and $Z(\mathcal{A})$ be the center of $\mathcal{A}$. A map $d : \mathcal{A} \to \mathcal{A}$ is called a derivation on $\mathcal{A}$ if $d(ab) = d(a)b + ad(b)$ holds for all $a, b \in \mathcal{A}$. Suppose that σ is an automorphism on $\mathcal{A}$. A map $d : \mathcal{A} \to \mathcal{A}$ is called σ-derivation on $\mathcal{A}$ if $d(ab) = d(a)b + \sigma(a)d(b)$ holds for all $a, b \in \mathcal{A}$. A map $d : \mathcal{A} \to \mathcal{A}$ is called σ-anti-derivation on $\mathcal{A}$ if $d(ab) = d(b)a + \sigma(b)d(a)$ holds for all $a, b \in \mathcal{A}$.

A *Morita context* consists of two unital $\mathcal{R}$-algebras $\mathcal{A}$ and $\mathcal{B}$, two bimodules $(\mathcal{A}, \mathcal{B})$-bimodule $\mathcal{M}$ and $(\mathcal{B}, \mathcal{A})$-bimodule $\mathcal{N}$, and two bimodule homomorphisms called the bilinear pairings $\xi_{\mathcal{MN}} : \mathcal{M} \underset{\mathcal{B}}{\otimes} \mathcal{N} \longrightarrow \mathcal{A}$ and $\Omega_{\mathcal{NM}} : \mathcal{N} \underset{\mathcal{A}}{\otimes} \mathcal{M} \longrightarrow \mathcal{B}$

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satisfying the following commutative diagrams:

$$\begin{array}{ccc} \mathcal{M} \underset{\mathcal{B}}{\otimes} \mathcal{N} \underset{\mathcal{A}}{\otimes} \mathcal{M} & \xrightarrow{\xi_{\mathcal{M}\mathcal{N}} \otimes I_{\mathcal{M}}} & \mathcal{A} \underset{\mathcal{A}}{\otimes} \mathcal{M} \\ \Big\downarrow{\scriptstyle I_{\mathcal{M}} \otimes \Omega_{\mathcal{N}\mathcal{M}}} & & \Big\downarrow{\scriptstyle \cong} \\ \mathcal{M} \underset{\mathcal{B}}{\otimes} \mathcal{B} & \xrightarrow{\cong} & \mathcal{M} \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{N} \underset{\mathcal{A}}{\otimes} \mathcal{M} \underset{\mathcal{B}}{\otimes} \mathcal{N} & \xrightarrow{\Omega_{\mathcal{N}\mathcal{M}} \otimes I_{\mathcal{N}}} & \mathcal{B} \underset{\mathcal{B}}{\otimes} \mathcal{N} \\ \Big\downarrow{\scriptstyle I_{\mathcal{N}} \otimes \xi_{\mathcal{M}\mathcal{N}}} & & \Big\downarrow{\scriptstyle \cong} \\ \mathcal{N} \underset{\mathcal{A}}{\otimes} \mathcal{A} & \xrightarrow{\cong} & \mathcal{N}. \end{array}$$

Let us write this Morita context as $(\mathcal{A}, \mathcal{B}, \mathcal{M}, \mathcal{N}, \xi_{\mathcal{M}\mathcal{N}}, \Omega_{\mathcal{N}\mathcal{M}})$. We refer the reader to [13] for the basic properties of Morita context. If $(\mathcal{A}, \mathcal{B}, \mathcal{M}, \mathcal{N}, \xi_{\mathcal{M}\mathcal{N}}, \Omega_{\mathcal{N}\mathcal{M}})$ is a Morita context, then the set

$$\begin{bmatrix} \mathcal{A} & \mathcal{M} \\ \mathcal{N} & \mathcal{B} \end{bmatrix} = \left\{ \begin{bmatrix} a & m \\ n & b \end{bmatrix} \middle| a \in \mathcal{A}, m \in \mathcal{M}, n \in \mathcal{N}, b \in \mathcal{B} \right\}$$

forms an $\mathcal{R}$-algebra under matrix addition and matrix-like multiplication, where at least one of the two bimodules $\mathcal{M}$ and $\mathcal{N}$ is distinct from zero. Such an $\mathcal{R}$-algebra is usually called a *generalized matrix algebra* of order 2 and is denoted by $\mathcal{G} = \mathcal{G}(\mathcal{A}, \mathcal{M}, \mathcal{N}, \mathcal{B}) = \begin{bmatrix} \mathcal{A} & \mathcal{M} \\ \mathcal{N} & \mathcal{B} \end{bmatrix}$. This kind of algebra was first introduced by Morita in [13], where the author investigated Morita duality theory of modules and its applications to Artinian algebras. All associative algebras with nontrivial idempotents are isomorphic to generalized matrix algebras. Most common examples of generalized matrix algebras are full matrix algebras over a unital algebra and triangular algebras [14, 15]. Also, if the bilinear pairings $\xi_{\mathcal{M}\mathcal{N}}$ and $\Omega_{\mathcal{N}\mathcal{M}}$ are zero, then $\mathcal{G}$ is called a trivial generalized matrix algebra and if $\mathcal{N} = 0$, then $\mathcal{G}$ is called a triangular algebra.

The center of $\mathcal{G}$ is

$$Z(\mathcal{G}) = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \middle| am = mb, na = bn \text{ for all } m \in \mathcal{M}, n \in \mathcal{N} \right\}.$$

Indeed $Z(\mathcal{G})$ is a set diagonal matrices $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$, where $a \in Z(\mathcal{A}), b \in Z(\mathcal{B})$ and $am = mb, na = bn$ for all $m \in \mathcal{M},\ n \in \mathcal{N}$. Also, in our case $\mathcal{M}$ is faithful left $\mathcal{A}$-module and right $\mathcal{B}$-module, then the condition $a \in Z(\mathcal{A}), b \in Z(\mathcal{B})$ is superfluous and can be removed. Define two natural projections $\pi_{\mathcal{A}} : \mathcal{G} \to \mathcal{A}$ and $\pi_{\mathcal{B}} : \mathcal{G} \to \mathcal{B}$ by $\pi_{\mathcal{A}} \begin{bmatrix} a & m \\ n & b \end{bmatrix} = a$ and $\pi_{\mathcal{B}} \begin{bmatrix} a & m \\ n & b \end{bmatrix} = b$. Moreover, $\pi_{\mathcal{A}}(Z(\mathcal{G})) \subseteq Z(\mathcal{A})$ and $\pi_{\mathcal{B}}(Z(\mathcal{G})) \subseteq Z(\mathcal{B})$ and there exists a unique algebraic isomorphism $\xi : \pi_{\mathcal{A}}(Z(\mathcal{G})) \to \pi_{\mathcal{B}}(Z(\mathcal{G}))$ such that $am = m\xi(a)$ and $na = \xi(a)n$ for all $a \in \pi_{\mathcal{A}}(Z(\mathfrak{A})), m \in \mathcal{M}$ and $n \in \mathcal{N}$.

Let $1_{\mathcal{A}}$ (resp.$1_{\mathcal{B}}$) be the identity of the algebra $\mathcal{A}$ (resp.$\mathcal{B}$) and let I be the identity of generalized matrix algebra $\mathcal{G}$, $e = \begin{bmatrix} 1_{\mathcal{A}} & 0 \\ 0 & 0 \end{bmatrix}$, $f = I - e = \begin{bmatrix} 0 & 0 \\ 0 & 1_{\mathcal{B}} \end{bmatrix}$ and $\mathcal{G}_{11} = e\mathcal{G}e$, $\mathcal{G}_{12} = e\mathcal{G}f$, $\mathcal{G}_{21} = f\mathcal{G}e$, $\mathcal{G}_{22} = f\mathcal{G}f$. Thus $\mathcal{G} = e\mathcal{G}e + e\mathcal{G}f + f\mathcal{G}e + f\mathcal{G}f = \mathcal{G}_{11} + \mathcal{G}_{12} + \mathcal{G}_{21} + \mathcal{G}_{22}$ where $\mathcal{G}_{11}$ is subalgebra of $\mathcal{G}$ isomorphic to $\mathcal{A}$, $\mathcal{G}_{22}$ is subalgebra of $\mathcal{G}$ isomorphic to $\mathcal{B}$, $\mathcal{G}_{12}$ is $(\mathcal{G}_{11}, \mathcal{G}_{22})$-bimodule isomorphic to $\mathcal{M}$ and $\mathcal{G}_{21}$ is $(\mathcal{G}_{22}, \mathcal{G}_{11})$-bimodule isomorphic to $\mathcal{N}$. Also, $\pi_{\mathcal{A}}(Z(\mathcal{G}))$ and $\pi_{\mathcal{B}}(Z(\mathcal{G}))$ are isomorphic to $eZ(\mathcal{G})e$ and $fZ(\mathcal{G})f$ respectively. Then there is an algebra isomorphisms $\xi : eZ(\mathcal{G})e \to fZ(\mathcal{G})f$ such that $am = m\xi(a)$ and $na = \xi(a)n$ for all $m \in e\mathcal{G}f$ and $n \in f\mathcal{G}e$.

There has been a great deal of work concerning characterizations of σ-derivations on rings. In the year 1957, Herstein [5] studied Jordan derivation on prime ring and proved that every Jordan derivation from a prime ring of characteristic not 2 into itself is a derivation. Brešar [2] proved that every Jordan derivation on a 2-torsion free semiprime ring is a derivation. Han and Wei [4] studied the Jordan (σ, τ)-derivation on triangular algebras $\mathcal{T}$ and proved that d is a Jordan (σ, τ)-derivation on $\mathcal{T}$ if and only if d is a (σ, τ)-derivation on $\mathcal{T}$. Several authors studied various kind of derivation on generalized matrix algebras [8, 9, 15]. Recently, Li and Wei [8] obtained the structure of derivation on generalized matrix algebra and Li, Wyk and Wei [9] proved that every Jordan derivation can be expressed as the sum of a derivation and an antiderivation on generalized matrix algebras.

Motivated by these studies our main purpose is to find out the structure of σ-derivation and Jordan σ- derivation on generalized matrix algebra. Also we show that every Jordan σ-derivation can be expressed as the sum of a σ-derivation and an anti skew-derivation on generalized matrix algebras.

2 Main Results

First of all we should mention some important results as follows:

Lemma 2.1. *[1, Propostion 2.1] Let $(\gamma, \delta, \mu, \nu, m_0, n_0)$ be a 6-tuple such that $\gamma : \mathcal{R} \to \mathcal{R}$ and $\delta : \mathcal{S} \to \mathcal{S}$ are endomorphisms $\mu : \mathcal{M} \to \mathcal{M}$ is $\gamma - \delta -$bimodule, $\nu : \mathcal{N} \to \mathcal{N}$ is a $\delta - \gamma -$bimodule automorphisms and $m_0 \in \mathcal{M}$ & $n_0 \in \mathcal{N}$ are fixed elements such that following conditions are satisfied:*

(i) $[m_0, \mathcal{N}] = 0$ *and* $(\mathcal{N}, m_0) = 0$,

(ii) $[\mathcal{M}, n_0] = 0$ *and* $(n_0, \mathcal{M}) = 0$, *(iii)* $[\mu(m), \nu(n)] = \gamma[m.n]$ *and* $(\nu(n), \mu(m)) = \delta(n, m)$.

Then the map $\sigma : \mathcal{G} \to \mathcal{G}$ *defined by*

$$\sigma \begin{bmatrix} a & m \\ n & b \end{bmatrix} = \begin{bmatrix} \gamma(a) & \gamma(a)m_0 - m_0\delta(b) + \mu(m) \\ n_0\gamma(a) - \delta(b)n_0 + \nu(n) & \delta(b) \end{bmatrix}$$

is a ring automorphism.

Now before the study of our main results, it is necessary to establish some substantial results as follows:

Proposition 2.1. *Let* $\mathcal{G} = \mathcal{G}(\mathcal{A}, \mathcal{M}, \mathcal{N}, \mathcal{B})$ *be a generalized matrix algebra over a commutative ring* $\mathcal{R}$ *with* $\mathcal{A}$ *and* $\mathcal{B}$ *have only trivial idempotents. An additive map* $\Phi_d : \mathcal{G} \to \mathcal{G}$ *is a* σ*-derivation on* $\mathcal{G}$ *if and only if* Φ_d *has the following form*

$$\Phi_d\left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\right)$$
$$= \begin{bmatrix} p_1(a) - \mu(m)n_0' + p_3(n) - m_0\delta(b)n_0' & \gamma(a)m_0' + r_2(m) - m_0'b \\ n_0'a + s_3(n) - \delta(b)n_0' & n_0\gamma(a)m_0' + q_2(m) + \nu(n)m_0' + q_4(b) \end{bmatrix}, \quad (\spadesuit)$$

where $a \in \mathcal{A}; b \in \mathcal{B}; m, m_0, m_0{}' \in \mathcal{M}; n, n_0, n_0{}' \in \mathcal{N}$ *and* $p_1 : \mathcal{A} \to \mathcal{A}, r_2 : \mathcal{M} \to \mathcal{M}$, $s_3 : \mathcal{N} \to \mathcal{N}, q_4 : \mathcal{B} \to \mathcal{B}, p_3 : \mathcal{N} \to \mathcal{A}, r_4 : \mathcal{B} \to \mathcal{M}, s_1 : \mathcal{A} \to \mathcal{N}, q_2 : \mathcal{M} \to \mathcal{B}$ *are* $\mathcal{R}$*-linear maps satisfying the following conditions:*

1. $p_1(a_1a_2) = p_1(a_1)a_2 + \gamma(a_1)p_1(a_2) + \gamma(a_1)m_0s_1(a_2)$ *and* $p_1(mn) = r_2(m)n + \mu(m)s_3(n)$;
2. $q_4(b_1b_2) = q_4(b_1)b_2 + \delta(b_1)q_4(b_2) - \delta(b_1)n_0r_4(b_2)$ *and* $q_4(nm) = s_3(n)m + \nu(n)r_2(m)$;
3. $r_2(am) = p_1(a)m + \gamma(a)r_2(m) + \gamma(a)m_0q_2(m)$ *and* $r_2(mb) = r_2(m)b + \mu(m)q_4(b)$;
4. $s_3(na) = s_3(n)a + \nu(n)p_1(a)$ *and* $s_3(bn) = q_4(b)n - \delta(b)n_0p_3(n) + \delta(b)s_3(n)$;
5. $p_3(n) = -m_0'n - m_0s_3(n)$ *and* $q_2(m) = n_0'm + n_0r_2(m)$.

Proof. Assume that σ-derivation takes the following form as

$$\Phi_d \begin{bmatrix} a & m \\ n & b \end{bmatrix} = \begin{bmatrix} p_1(a) + p_2(m) + p_3(n) + p_4(b) & r_1(a) + r_2(m) + r_3(n) + r_4(b) \\ s_1(a) + s_2(m) + s_3(n) + s_4(b) & q_1(a) + q_2(m) + q_3(n) + q_4(b) \end{bmatrix}$$

where $p_1 : \mathcal{A} \to \mathcal{A}, p_2 : \mathcal{M} \to \mathcal{A}, p_3 : \mathcal{N} \to \mathcal{A}, p_4 : \mathcal{B} \to \mathcal{A}$; $r_1 : \mathcal{A} \to \mathcal{M}, r_2 : \mathcal{M} \to \mathcal{M}, r_3 : \mathcal{N} \to \mathcal{M}, r_4 : \mathcal{B} \to \mathcal{M}$; $s_1 : \mathcal{A} \to \mathcal{N}, s_2 : \mathcal{M} \to \mathcal{N}, s_3 : \mathcal{N} \to$

$\mathcal{N}, s_4 : \mathcal{B} \to \mathcal{N}$ and $q_1 : \mathcal{A} \to \mathcal{B}, q_2 : \mathcal{M} \to \mathcal{B}, q_3 : \mathcal{N} \to \mathcal{B}, q_4 : \mathcal{B} \to \mathcal{B}$ are $\mathcal{R}$-linear maps. As Φ_d is the σ-derivation with automorphism σ defined by $\Phi_d(G_1G_2) = \Phi_d(G_1)G_2 + \sigma(G_1)\Phi_d(G_2)$ for all $G_1, G_2 \in \mathcal{G}$. Now we assume that $G_1 = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$ and $G_2 = \begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}$ and applying Lemma 2.1

$$\begin{aligned}
&\Phi_d\left(\begin{bmatrix} 0 & am \\ 0 & 0 \end{bmatrix}\right) \\
&\quad = \Phi_d\left(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\right)\begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix} + \sigma\left(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\right)\Phi_d\left(\begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}\right) \\
&\begin{bmatrix} p_2(am) & r_2(am) \\ s_2(am) & q_2(am) \end{bmatrix} \\
&\quad = \begin{bmatrix} p_1(a) & r_1(a) \\ s_1(a) & q_1(a) \end{bmatrix}\begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix} \\
&\qquad + \begin{bmatrix} \gamma(a) & \gamma(a)m_0 \\ n_0\gamma(a) & 0 \end{bmatrix}\begin{bmatrix} p_2(m) & r_2(m) \\ s_2(m) & q_2(m) \end{bmatrix} \\
&\quad = \begin{bmatrix} 0 & p_1(a)m \\ 0 & s_1(a)m \end{bmatrix} \\
&\qquad + \begin{bmatrix} \gamma(a)p_2(m) + \gamma(a)m_0s_2(m) & \gamma(a)r_2(m) + \gamma(a)m_0q_2(m) \\ n_0\gamma(a)p_2(m) & n_0\gamma(a)r_2(m) \end{bmatrix}.
\end{aligned}$$

On comparing both sides, we get

$$\begin{aligned}
p_2(am) &= \gamma(a)p_2(m) + \gamma(a)m_0s_2(m), \\
r_2(am) &= p_1(a)m + \gamma(a)r_2(m) + \gamma(a)m_0q_2(m), \\
s_2(am) &= n_0\gamma(a)p_2(m), \\
q_2(am) &= s_1(a)m + n_0\gamma(a)r_2(m).
\end{aligned}$$

Substitute $a = 1$, we get $m_0s_2(m) = 0, p_1(1)m + m_0q_2(m) = 0, s_2(m) = n_0p_2(m)$ and $q_2(m) = s_1(1)m + n_0r_2(m)$. In a similar way, on assuming $G_1 = \begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}$ and $G_2 = \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}$, we obtain that $p_2(mb) = \mu(m)s_4(b), r_2(mb) = r_2(m)b + \mu(m)q_4(b)$, $s_2(mb) = 0$ and $q_2(mb) = q_2(m)b$. If $b = 1$, then we get $p_2(m) = -\mu(m)n'_0$, where $s_4(1) = -n'_0, \mu(m)q_4(1) = 0$ and $s_2(m) = 0$.

Now let us take $G_1 = \begin{bmatrix} 0 & 0 \\ n & 0 \end{bmatrix}$ and $G_2 = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$, we have

$$\begin{aligned}
&\Phi_d\left(\begin{bmatrix} 0 & 0 \\ na & 0 \end{bmatrix}\right) \\
&\quad = \Phi_d\left(\begin{bmatrix} 0 & 0 \\ n & 0 \end{bmatrix}\right)\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} + \sigma\left(\begin{bmatrix} 0 & 0 \\ n & 0 \end{bmatrix}\right)\Phi_d\left(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\right) \\
&\begin{bmatrix} p_3(na) & r_3(na) \\ s_3(na) & q_3(na) \end{bmatrix} \\
&\quad = \begin{bmatrix} p_3(n) & r_3(n) \\ s_3(n) & q_3(n) \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \nu(n) & 0 \end{bmatrix}\begin{bmatrix} p_1(a) & r_1(a) \\ s_1(a) & q_1(a) \end{bmatrix} \\
&\quad = \begin{bmatrix} p_3(n)a & 0 \\ s_3(n)a & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \nu(n)p_1(a) & \nu(n)r_1(a) \end{bmatrix}.
\end{aligned}$$

On equating both sides, $p_3(na) = p_3(n)a, r_3(na) = 0$, $s_3(na) = s_3(n)a + \nu(n)p_1(a)$ and $q_3(na) = \nu(n)r_1(a)$. Substitute $a = 1$, we get $r_3(n) = 0$, $\nu(n)p_1(1) = 0$ and $q_3(n) = \nu(n)m'_0$, where $r_1(1) = m'_0$. Similarly, on taking $G_1 = \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}$ and $G_2 = \begin{bmatrix} 0 & 0 \\ n & 0 \end{bmatrix}$, we have

$$\begin{aligned}
p_3(bn) &= r_4(b)n - m_0\delta(b)s_3(n), \\
r_3(bn) &= -m_0\delta(b)q_3(n), \\
s_3(bn) &= q_4(b)n - \delta(b)n_0p_3(n) + \delta(b)s_3(n), \\
q_3(bn) &= -\delta(b)n_0r_3(n) + \delta(b)q_3(n).
\end{aligned}$$

Substitute $b = 1$, we get $p_3(n) = r_4(1)n - m_0s_3(n), r_3(n) = -m_0q_3(n), q_4(1)n = n_0p_3(n)$ and $-n_0r_3(n) = 0$.

Suppose that $G_1 = \begin{bmatrix} a_1 & 0 \\ 0 & 0 \end{bmatrix}$ and $G_2 = \begin{bmatrix} a_2 & 0 \\ 0 & 0 \end{bmatrix}$, we obtain that

$$\begin{aligned}
&\Phi_d\left(\begin{bmatrix} a_1a_2 & 0 \\ 0 & 0 \end{bmatrix}\right) \\
&\quad = \Phi_d\left(\begin{bmatrix} a_1 & 0 \\ 0 & 0 \end{bmatrix}\right)\begin{bmatrix} a_2 & 0 \\ 0 & 0 \end{bmatrix} + \sigma\left(\begin{bmatrix} a_1 & 0 \\ 0 & 0 \end{bmatrix}\right)\Phi_d\left(\begin{bmatrix} a_2 & 0 \\ 0 & 0 \end{bmatrix}\right) \\
&\begin{bmatrix} p_1(a_1a_2) & r_1(a_1a_2) \\ s_1(a_1a_2) & q_1(a_1a_2) \end{bmatrix} \\
&\quad = \begin{bmatrix} p_1(a_1) & r_1(a_1) \\ s_1(a_1) & q_1(a_1) \end{bmatrix}\begin{bmatrix} a_2 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \gamma(a_1) & \gamma(a_1)m_0 \\ n_0\gamma(a_1) & 0 \end{bmatrix}\begin{bmatrix} p_1(a_2) & r_1(a_2) \\ s_1(a_2) & q_1(a_2) \end{bmatrix} \\
&\quad = \begin{bmatrix} p_1(a_1)a_2 + \gamma(a_1)p_1(a_2) + \gamma(a_1)m_0s_1(a_2) & \gamma(a_1)r_1(a_2) + \gamma(a_1)m_0q_1(a_2) \\ s_1(a_1)a_2 + n_0\gamma(a_1)p_1(a_2) & n_0\gamma(a_1)r_1(a_2) \end{bmatrix}.
\end{aligned}$$

On comparing both sides, we get

$$\begin{aligned}
p_1(a_1a_2) &= p_1(a_1)a_2 + \gamma(a_1)p_1(a_2) + \gamma(a_1)m_0s_1(a_2),\\
r_1(a_1a_2) &= \gamma(a_1)r_1(a_2) + \gamma(a_1)m_0q_1(a_2),\\
s_1(a_1a_2) &= s_1(a_1)a_2 + n_0\gamma(a_1)p_1(a_2),\\
q_1(a_1a_2) &= n_0\gamma(a_1)r_1(a_2).
\end{aligned}$$

Put $a_1 = 1, a_2 = a$, we get $m_0s_1(a) = p_1(1)a, m_0q_1(a) = 0, s_1(a) = s_1(1)a + n_0p_1(a)$ and $q_1(a) = n_0r_1(a)$. Again, put $a_1 = a, a_2 = 1$, we have $\gamma(a)p_1(1) + \gamma(a)m_0s_1(1) = 0, r_1(a) = \gamma(a)r_1(1) + \gamma(a)m_0q_1(1), n_0\gamma(a)p_1(1) = 0$ and $q_1(a) = n_0\gamma(a)r_1(1)$. Similarly, suppose that $G_1 = \begin{bmatrix} 0 & 0 \\ 0 & b_1 \end{bmatrix}$ and $G_2 = \begin{bmatrix} 0 & 0 \\ 0 & b_2 \end{bmatrix}$, we get

$$\begin{aligned}
p_4(b_1b_2) &= -m_0\delta(b_1)s_4(b_2),\\
r_4(b_1b_2) &= r_4(b_1)b_2 - m_0\delta(b_1)q_4(b_2),\\
s_4(b_1b_2) &= -\delta(b_1)n_0p_4(b_2) + \delta(b_1)s_4(b_2),\\
q_4(b_1b_2) &= q_4(b_1)b_2 - \delta(b_1)n_0r_4(b_2) + \delta(b_1)q_4(b_2).
\end{aligned}$$

Put $b_1 = 1, b_2 = b$, it follows that $p_4(b) = -m_0s_4(b), r_4(b) = r_4(1)b - m_0q_4(b), -n_0p_4(b) = 0$ and $q_4(1)b = n_0r_4(b)$. Also, if $b_1 = b, b_2 = 1$, then $p_4(b) = -m_0\delta(b)s_4(1), -m_0\delta(b)q_4(1) = 0, s_4(b) = -\delta(b)n_0p_4(1) + \delta(b)s_4(1)$ and $-\delta(b)n_0r_4(1) + \delta(b)q_4(1) = 0$.

Now if $G_1 = \begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}$ and $G_2 = \begin{bmatrix} 0 & 0 \\ n & 0 \end{bmatrix}$, then we have

$$\begin{aligned}
&\Phi_d\left(\begin{bmatrix} mn & 0 \\ 0 & 0 \end{bmatrix}\right)\\
&= \Phi_d\left(\begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}\right)\begin{bmatrix} 0 & 0 \\ n & 0 \end{bmatrix} + \sigma\left(\begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}\right)\Phi_d\left(\begin{bmatrix} 0 & 0 \\ n & 0 \end{bmatrix}\right)\\
&\begin{bmatrix} p_1(mn) & r_1(mn) \\ s_1(mn) & q_1(mn) \end{bmatrix}\\
&= \begin{bmatrix} p_2(m) & r_2(m) \\ s_2(m) & q_2(m) \end{bmatrix}\begin{bmatrix} 0 & 0 \\ n & 0 \end{bmatrix} + \begin{bmatrix} 0 & \mu(m) \\ 0 & 0 \end{bmatrix}\begin{bmatrix} p_3(n) & r_3(n) \\ s_3(n) & q_3(n) \end{bmatrix}\\
&= \begin{bmatrix} r_2(m)n & 0 \\ q_2(m)n & 0 \end{bmatrix} + \begin{bmatrix} \mu(m)s_3(n) & \mu(m)q_3(n) \\ 0 & 0 \end{bmatrix}.
\end{aligned}$$

This leads to $p_1(mn) = r_2(m)n + \mu(m)s_3(n), r_1(mn) = \mu(m)q_3(n), s_1(mn) = q_2(m)n, q_1(mn) = 0$. Follow similarly $G_1 = \begin{bmatrix} 0 & 0 \\ n & 0 \end{bmatrix}$ and $G_2 = \begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}$, we arrive at $p_4(nm) = 0, r_4(nm) = p_3(n)m, s_4(nm) = \nu(n)p_2(m), q_4(nm) = s_3(n)m + \nu(n)r_2(m)$.

Again, suppose that $G_1 = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$ and $G_2 = \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}$

$$\begin{aligned}
&\Phi_d\left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right) \\
&= \Phi_d\left(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\right)\begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix} + \sigma\left(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\right)\Phi_d\left(\begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}\right) \\
&\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} p_1(a) & r_1(a) \\ s_1(a) & q_1(a) \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix} + \begin{bmatrix} \gamma(a) & \gamma(a)m_0 \\ n_0\gamma(a) & 0 \end{bmatrix}\begin{bmatrix} p_4(b) & r_4(b) \\ s_4(b) & q_4(b) \end{bmatrix} \\
&= \begin{bmatrix} 0 & r_1(a)b \\ 0 & q_1(a)b \end{bmatrix} + \begin{bmatrix} \gamma(a)p_4(b) + \gamma(a)m_0s_4(b) & \gamma(a)r_4(b) + \gamma(a)m_0q_4(b) \\ n_0\gamma(a)p_4(b) & n_0\gamma(a)r_4(b) \end{bmatrix}.
\end{aligned}$$

On comparing both sides we get

$$\begin{aligned}
\gamma(a)p_4(b) + \gamma(a)m_0s_4(b) &= 0, \\
\gamma(a)r_4(b) + r_1(a)b &= 0, \\
n_0\gamma(a)p_4(b) &= 0, \\
q_1(a)b + n_0\gamma(a)r_4(b) &= 0.
\end{aligned}$$

Using $a = 1$, we get $p_4(b) + m_0s_4(b) = 0, r_4(b) + r_1(1)b = 0, n_0p_4(b) = 0$ and $q_1(1)b + n_0r_4(b) = 0$. Also, substitute $b = 1$, we get $\gamma(a)p_4(1) + \gamma(a)m_0s_4(1) = 0, \gamma(a)r_4(1) + r_1(a) = 0, n_0\gamma(a)p_4(1) = 0$ and $q_1(a) + n_0\gamma(a)r_4(1) = 0$. Further, on taking $a = 1$ & $b = 1$, $r_1(1) = -r_4(1) = -m_0'$.

On following similar steps with $G_1 = \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}$ and $G_2 = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$, we get

$$\begin{aligned}
p_4(b)a - m_0\delta(b)s_1(a) &= 0, \\
-m_0\delta(b)q_1(a) &= 0, \\
s_4(b)a - \delta(b)n_0p_1(a) + \delta(b)s_1(a) &= 0, \\
-\delta(b)n_0r_1(a) + \delta(b)q_1(a) &= 0.
\end{aligned}$$

Substitute $b = 1$, we get $p_4(1) - m_0s_1(a) = 0, -m_0q_1(a) = 0, s_4(1)a + s_1(a) = 0$ and $n_0r_1(a) + q_1(a) = 0$. Again when $a = 1$, we have $p_4(b) - m_0\delta(b)s_1(1) = 0, -m_0\delta(b)q_1(1) = 0, s_4(b) + \delta(b)s_1(1) = 0$ and $-\delta(b)n_0r_1(1) + \delta(b)q_1(1) = 0$. Also for $a = 1$ & $b = 1$, $s_4(1) = -s_1(1) = -n_0'$.

If Φ_d has form (♠) and satisfies condition (1) to (5), the assertion that Φ_d is a σ-derivation on $\mathcal{G}$ will follow from direct computations. $\square$

Proposition 2.2. *Let $\mathcal{G} = \mathcal{G}(\mathcal{A}, \mathcal{M}, \mathcal{N}, \mathcal{B})$ be a generalized matrix algebra over a commutative ring $\mathcal{R}$ with $\mathcal{A}$ and $\mathcal{B}$ have only trivial idempotents. An additive map $\Phi : \mathcal{G} \to \mathcal{G}$ is an anti-σ-derivation on $\mathcal{G}$ if and only if Φ_{ad} has the following form*

$$\Phi_{ad}\left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\right) = \begin{bmatrix} p_1(a) + p_2(m) - m_0\delta(b)n_0' & \gamma(a)m_0' + r_2(m) + r_3(n) + m_0'b \\ n_0'a + s_2(m) + s_3(n) - \delta(b)n_0' & n_0\gamma(a)m_0' + q_3(n) + q_4(b) \end{bmatrix}, \tag{$\star$}$$

where $a \in \mathcal{A}; b \in \mathcal{B}; m, m_0, m_0{}' \in \mathcal{M}; n, n_0, n_0{}' \in \mathcal{N}$ and $p_1 : \mathcal{A} \to \mathcal{A}, r_2 : \mathcal{M} \to \mathcal{M}$, $s_3 : \mathcal{N} \to \mathcal{N}, q_4 : \mathcal{B} \to \mathcal{B}, p_3 : \mathcal{N} \to \mathcal{A}, r_4 : \mathcal{B} \to \mathcal{M}, s_1 : \mathcal{A} \to \mathcal{N}, q_2 : \mathcal{M} \to \mathcal{B}$ are $\mathcal{R}$-linear maps satisfying the following conditions:

1. $[\gamma(a_1), \gamma(a_2)]m_0' = 0, [\delta(b_1), \delta(b_2)]n_0' = 0$ *and* $\nu(n)m_0' = 0, \mu(m)n_0' = 0$,
2. $s_2(am) = s_2(m)a, s_2(mb) = \delta(b)s_2(m), \mu(m_2)s_2(m_1) = 0$ *and* $s_2(m_2)m_1 = 0$,
3. $r_3(na) = \gamma(a)r_3(n), r_3(bn) = r_3(n)b, r_3(n_2)n_1 = 0$ *and* $\nu(n_2)r_3(n_1) = 0$,
4. $r_2(m) = p_4(1)m = \mu(m)q_1(1)$ *and* $s_3(n) = \nu(n)p_4(1) = q_1(1)n$,
5. $p_2(m) = -m_0s_2(m)$, *and* $q_3(n) = n_0r_3(n)$.

Proof. Assume that anti σ-derivation has the following form

$$\Phi_{ad}\begin{bmatrix} a & m \\ n & b \end{bmatrix} = \begin{bmatrix} p_1(a) + p_2(m) + p_3(n) + p_4(b) & r_1(a) + r_2(m) + r_3(n) + r_4(b) \\ s_1(a) + s_2(m) + s_3(n) + s_4(b) & q_1(a) + q_2(m) + q_3(n) + q_4(b) \end{bmatrix},$$

where $p_1 : \mathcal{A} \to \mathcal{A}, p_2 : \mathcal{M} \to \mathcal{A}, p_3 : \mathcal{N} \to \mathcal{A}, p_4 : \mathcal{B} \to \mathcal{A}$; $r_1 : \mathcal{A} \to \mathcal{M}, r_2 : \mathcal{M} \to \mathcal{M}, r_3 : \mathcal{N} \to \mathcal{M}, r_4 : \mathcal{B} \to \mathcal{M}$; $s_1 : \mathcal{A} \to \mathcal{N}, s_2 : \mathcal{M} \to \mathcal{N}, s_3 : \mathcal{N} \to \mathcal{N}, s_4 : \mathcal{B} \to \mathcal{N}$ and $q_1 : \mathcal{A} \to \mathcal{B}$,$q_2 : \mathcal{M} \to \mathcal{B}, q_3 : \mathcal{N} \to \mathcal{B}, q_4 : \mathcal{B} \to \mathcal{B}$ are $\mathcal{R}$-linear maps. As Φ_{ad} is the anti-σ-derivation with automorphism σ defined by $\Phi_{ad}(G_1G_2) = \Phi_{ad}(G_2)G_1 + \sigma(G_2)\Phi_{ad}(G_1)$ for all $G_1, G_2 \in \mathcal{G}$. Now we assume that $G_1 = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$ and $G_2 = \begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}$ and applying Lemma 2.1

$$\begin{aligned}
&\Phi_{ad}\left(\begin{bmatrix} 0 & am \\ 0 & 0 \end{bmatrix}\right) \\
&= \Phi_{ad}\left(\begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}\right)\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} + \sigma\left(\begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}\right)\Phi_{ad}\left(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\right) \\
&\begin{bmatrix} p_2(am) & r_2(am) \\ s_2(am) & q_2(am) \end{bmatrix} \\
&= \begin{bmatrix} p_2(m) & r_2(m) \\ s_2(m) & q_2(m) \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & \mu(m) \\ 0 & 0 \end{bmatrix}\begin{bmatrix} p_1(a) & r_1(a) \\ s_1(a) & q_1(a) \end{bmatrix} \\
&= \begin{bmatrix} p_2(m)a & 0 \\ s_2(m)a & 0 \end{bmatrix} + \begin{bmatrix} \mu(m)s_1(a) & \mu(m)q_1(a) \\ 0 & 0 \end{bmatrix}.
\end{aligned}$$

On comparing both sides we get $p_2(am) = p_2(m)a + \mu(m)s_1(a), r_2(am) = \mu(m)q_1(a), s_2(am) = s_2(m)a$ and $q_2(am) = 0$. On putting $a = 1$, we get $\mu(m)n_0' = 0$, where $n_0' = s_1(1)$ and$r_2(m) = \mu(m)q_1(1), q_2(m) = 0$. In a similar way on assuming $G_1 = \begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}$ and $G_2 = \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}$, we obtain that

$$\begin{aligned}
p_2(mb) &= -m_0\delta(b)s_2(m), \\
r_2(mb) &= -m_0\delta(b)q_2(m) + p_4(b)m, \\
s_2(mb) &= -\delta(b)n_0p_2(m) + \delta(b)s_2(m), \\
q_2(mb) &= s_4(b)m - \delta(b)n_0r_2(m) + \delta(b)q_2(m).
\end{aligned}$$

On substituting $b = 1$, we get $p_2(m) = -m_0s_2(m), r_2(m) = p_4(1)m, 0 = -n_0p_2(m)$ and $0 = s_4(1)m - n_0r_2(m)$. This implies that $s_2(mb) = \delta(b)s_2(m)$

Now let us take $G_1 = \begin{bmatrix} 0 & 0 \\ n & 0 \end{bmatrix}$ and $G_1 = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$, we have

$$\begin{aligned}
&\Phi_{ad}\left(\begin{bmatrix} 0 & 0 \\ na & 0 \end{bmatrix}\right) \\
&= \Phi_{ad}\left(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\right)\begin{bmatrix} 0 & 0 \\ n & 0 \end{bmatrix} + \sigma\left(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\right)\Phi_{ad}\left(\begin{bmatrix} 0 & 0 \\ n & 0 \end{bmatrix}\right) \\
&\begin{bmatrix} p_3(na) & r_3(na) \\ s_3(na) & q_3(na) \end{bmatrix} \\
&= \begin{bmatrix} p_1(a) & r_1(a) \\ s_1(a) & q_1(a) \end{bmatrix}\begin{bmatrix} 0 & 0 \\ n & 0 \end{bmatrix} + \begin{bmatrix} \gamma(a) & \gamma(a)m_0 \\ n_0\gamma(a) & 0 \end{bmatrix}\begin{bmatrix} p_3(n) & r_3(n) \\ s_3(n) & q_3(n) \end{bmatrix} \\
&= \begin{bmatrix} r_1(a)n & 0 \\ q_1(a)n & 0 \end{bmatrix} \\
&\quad + \begin{bmatrix} \gamma(a)p_3(n) + \gamma(a)m_0s_3(n) & \gamma(a)r_3(n) + \gamma(a)m_0q_3(n) \\ n_0\gamma(a)p_3(n) & n_0\gamma(a)r_3(n) \end{bmatrix}.
\end{aligned}$$

On equating both sides,

$$\begin{aligned}
p_3(na) &= r_1(a)n + \gamma(a)p_3(n) + \gamma(a)m_0s_3(n), \\
r_3(na) &= \gamma(a)r_3(n) + \gamma(a)m_0q_3(n), \\
s_3(na) &= q_1(a)n + n_0\gamma(a)p_3(n), \\
q_3(na) &= n_0\gamma(a)r_3(n).
\end{aligned}$$

Substitute $a = 1$, we get $0 = r_1(1)n + m_0s_3(n), 0 = m_0q_3(n), s_3(n) = q_1(1)n + n_0p_3(n)$ and $q_3(n) = n_0r_3(n)$. This leads to $r_3(na) = \gamma(a)r_3(n)$. Similarly, on taking $G_1 = \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}$ and $G_2 = \begin{bmatrix} 0 & 0 \\ n & 0 \end{bmatrix}$, we have $p_3(bn) = 0, r_3(bn) = r_3(n)b, s_3(bn) = \nu(n)p_4(b)$, and $q_3(bn) = q_3(n)b + \nu(n)r_4(b)$. Put $b = 1$, we get $p_3(n) = 0, s_3(n) = \nu(n)p_4(1)$ and $0 = \nu(n)r_4(1)$.

Suppose that $G_1 = \begin{bmatrix} a_1 & 0 \\ 0 & 0 \end{bmatrix}$ and $G_2 = \begin{bmatrix} a_2 & 0 \\ 0 & 0 \end{bmatrix}$, we obtain that

$$\begin{aligned}
&\Phi_{ad}\left(\begin{bmatrix} a_1a_2 & 0 \\ 0 & 0 \end{bmatrix}\right) \\
&= \Phi_{ad}\left(\begin{bmatrix} a_2 & 0 \\ 0 & 0 \end{bmatrix}\right)\begin{bmatrix} a_1 & 0 \\ 0 & 0 \end{bmatrix} + \sigma\left(\begin{bmatrix} a_2 & 0 \\ 0 & 0 \end{bmatrix}\right)\Phi_{ad}\left(\begin{bmatrix} a_1 & 0 \\ 0 & 0 \end{bmatrix}\right) \\
&\begin{bmatrix} p_1(a_1a_2) & r_1(a_1a_2) \\ s_1(a_1a_2) & q_1(a_1a_2) \end{bmatrix} \\
&= \begin{bmatrix} p_1(a_2) & r_1(a_2) \\ s_1(a_2) & q_1(a_2) \end{bmatrix}\begin{bmatrix} a_1 & 0 \\ 0 & 0 \end{bmatrix} \\
&\quad + \begin{bmatrix} \gamma(a_2) & \gamma(a_2)m_0 \\ n_0\gamma(a_2) & 0 \end{bmatrix}\begin{bmatrix} p_1(a_1) & r_1(a_1) \\ s_1(a_1) & q_1(a_1) \end{bmatrix}
\end{aligned}$$

On comparing both sides, we get

$$\begin{aligned}
p_1(a_1a_2) &= p_1(a_2)a_1 + \gamma(a_2)p_1(a_1) + \gamma(a_2)m_0s_1(a_1), \\
r_1(a_1a_2) &= \gamma(a_2)r_1(a_1) + \gamma(a_2)m_0q_1(a_1), \\
s_1(a_1a_2) &= s_1(a_2)a_1 + n_0\gamma(a_2)p_1(a_1), \\
q_1(a_1a_2) &= n_0\gamma(a_2)r_1(a_1).
\end{aligned}$$

Put $a_2 = 1, a_1 = a$, we get $m_0s_1(a) = p_1(1)a, m_0q_1(a) = 0, s_1(a) = s_1(1)a + n_0p_1(a)$ and $q_1(a) = n_0r_1(a)$. Again put $a_2 = a, a_1 = 1$, we have $\gamma(a)p_1(1) + \gamma(a)m_0s_1(1) = 0, r_1(a) = \gamma(a)r_1(1) + \gamma(a)m_0q_1(1), n_0\gamma(a)p_1(1) = 0$ and $q_1(a) = n_0\gamma(a)r_1(1)$. This leads to $r_1(a_1a_2) = \gamma(a_2)r_1(a_1)$ and from here we can conclude that $[\gamma(a_1), \gamma(a_2)]m_0' = 0$. Similarly,suppose that $G_1 = \begin{bmatrix} 0 & 0 \\ 0 & b_1 \end{bmatrix}$ and $G_1 = \begin{bmatrix} 0 & 0 \\ 0 & b_2 \end{bmatrix}$, we get

$$\begin{aligned}
p_4(b_1b_2) &= -m_0\delta(b_2)s_4(b_1), \\
r_4(b_1b_2) &= r_4(b_2)b_1 - m_0\delta(b_2)q_4(b_1), \\
s_4(b_1b_2) &= -\delta(b_2)n_0p_4(b_1) + \delta(b_2)s_4(b_1), \\
q_4(b_1b_2) &= q_4(b_2)b_1 - \delta(b_2)n_0r_4(b_1) + \delta(b_2)q_4(b_1).
\end{aligned}$$

Substitute $b_2 = 1, b_1 = b$, it follows that $p_4(b) = -m_0s_4(b), r_4(b) = r_4(1)b - m_0q_4(b), -n_0p_4(b) = 0$ and $q_4(1)b = n_0r_4(b)$. Also, if $b_2 = b, b_1 = 1$, then $p_4(b) = -m_0\delta(b)s_4(1), -m_0\delta(b)q_4(1) = 0, s_4(b) = -\delta(b)n_0p_4(1) + \delta(b)s_4(1)$ and $-\delta(b)n_0r_4(1) + \delta(b)q_4(1) = 0$. This implies that $s_4(b_1b_2) = \delta(b_2)s_4(b_1)$ and hence $m_0'[\delta(b_1), \delta(b_2)] = 0$.

Now if $G_1 = \begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}$ and $G_2 = \begin{bmatrix} 0 & 0 \\ n & 0 \end{bmatrix}$, then we have

$$\begin{aligned}
&\Phi_{ad}\left(\begin{bmatrix} mn & 0 \\ 0 & 0 \end{bmatrix}\right)\\
&\quad = \Phi_{ad}\left(\begin{bmatrix} 0 & 0 \\ n & 0 \end{bmatrix}\right)\begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix} + \sigma\left(\begin{bmatrix} 0 & 0 \\ n & 0 \end{bmatrix}\right)\Phi_{ad}\left(\begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}\right)\\
&\begin{bmatrix} p_1(mn) & r_1(mn) \\ s_1(mn) & q_1(mn) \end{bmatrix}\\
&\quad = \begin{bmatrix} p_3(n) & r_3(n) \\ s_3(n) & q_3(n) \end{bmatrix}\begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \nu(n) & 0 \end{bmatrix}\begin{bmatrix} p_2(m) & r_2(m) \\ s_2(m) & q_2(m) \end{bmatrix}\\
&\quad = \begin{bmatrix} 0 & p_3(n)m \\ 0 & s_3(n)m \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \nu(n)p_2(m) & \nu(n)r_2(m) \end{bmatrix}
\end{aligned}$$

This leads to $p_1(mn) = 0, r_1(mn) = p_3(n)m, s_1(mn) = \nu(n)p_2(m), q_1(mn) = s_3(n)m + \nu(n)r_2(m)$. Follow similarly $G_1 = \begin{bmatrix} 0 & 0 \\ n & 0 \end{bmatrix}$ and $G_2 = \begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}$, we arrive at $p_4(nm) = r_2(m)n + \mu(m)s_3(n)$,
$r_4(nm) = \mu(m)q_3(n)$,$s_4(nm) = q_2(m)n$ and $q_4(nm) = 0$.

Again, suppose that $G_1 = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$ and $G_2 = \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}$

$$\begin{aligned}
&\Phi_{ad}\left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right)\\
&\quad = \Phi_{ad}\left(\begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}\right)\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} + \sigma\left(\begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}\right)\Phi_{ad}\left(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\right)\\
&\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\\
&\quad = \begin{bmatrix} p_4(b) & r_4(b) \\ s_4(b) & q_4(b) \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -m_0\delta(b) \\ -\delta(b)n_0 & \delta(b) \end{bmatrix}\begin{bmatrix} p_1(a) & r_1(a) \\ s_1(a) & q_1(a) \end{bmatrix}\\
&\quad = \begin{bmatrix} p_4(b)a & 0 \\ s_4(b)a & 0 \end{bmatrix}\\
&\qquad + \begin{bmatrix} -m_0\delta(b)s_1(a) & -m_0\delta(b)q_1(a) \\ -\delta(b)n_0p_1(a) + \delta(b)s_1(a) & -\delta(b)n_0r_1(a) + \delta(b)q_1(a) \end{bmatrix}
\end{aligned}$$

On comparing both sides we get $p_4(b)a - m_0\delta(b)s_1(a) = 0, -m_0\delta(b)q_1(a) = 0, s_4(b)a - \delta(b)n_0p_1(a) + \delta(b)s_1(a) = 0$ and $-\delta(b)n_0r_1(a) + \delta(b)q_1(a) = 0$. Substitute $b = 1$, we get $p_4(1) - m_0s_1(a) = 0, -m_0q_1(a) = 0, s_4(1)a + s_1(a) = 0$ and $n_0r_1(a) + q_1(a) = 0$. Again when $a = 1$, we have $p_4(b) - m_0\delta(b)s_1(1) =$

$0, -m_0\delta(b)q_1(1) = 0, s_4(b) + \delta(b)s_1(1) = 0$ and $-\delta(b)n_0r_1(1) + \delta(b)q_1(1) = 0$. Also for $a = 1$ & $b = 1$, $s_4(1) = -s_1(1) = -n_0'$.

On following similar steps with $G_1 = \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}$ and $G_2 = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$, we get $\gamma(a)p_4(b) + \gamma(a)m_0s_4(b) = 0, \gamma(a)r_4(b) + r_1(a)b = 0, n_0\gamma(a)p_4(b) = 0$ and $q_1(a)b + n_0\gamma(a)r_4(b) = 0$. Using $a = 1$, we get $p_4(b) + m_0s_4(b) = 0, r_4(b) + r_1(1)b = 0, n_0p_4(b) = 0$ and $q_1(1)b + n_0r_4(b) = 0$. Also, substitute $b = 1$, we get $\gamma(a)p_4(1) + \gamma(a)m_0s_4(1) = 0, \gamma(a)r_4(1) + r_1(a) = 0, n_0\gamma(a)p_4(1) = 0$ and $q_1(a) + n_0\gamma(a)r_4(1) = 0$. Further, on taking $a = 1$ & $b = 1$, $r_1(1) = -r_4(1) = -m_0'$.

Now, suppose that $G_1 = \begin{bmatrix} 0 & m_1 \\ 0 & 0 \end{bmatrix}$ and $G_2 = \begin{bmatrix} 0 & m_2 \\ 0 & 0 \end{bmatrix}$

$$\begin{aligned}
&\Phi_{ad}\left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right) \\
&= \Phi_{ad}\left(\begin{bmatrix} 0 & m_2 \\ 0 & 0 \end{bmatrix}\right)\begin{bmatrix} 0 & m_1 \\ 0 & 0 \end{bmatrix} + \sigma\left(\begin{bmatrix} 0 & m_2 \\ 0 & 0 \end{bmatrix}\right)\Phi_{ad}\left(\begin{bmatrix} 0 & m_1 \\ 0 & 0 \end{bmatrix}\right)
\end{aligned}$$

$$\begin{aligned}
&\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} p_2(m_2) & r_2(m_2) \\ s_2(m_2) & q_2(m_2) \end{bmatrix}\begin{bmatrix} 0 & m_1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & \mu(m_2) \\ 0 & 0 \end{bmatrix}\begin{bmatrix} p_2(m_1) & r_2(m_1) \\ s_2(m_1) & q_2(m_1) \end{bmatrix} \\
&= \begin{bmatrix} 0 & p_2(m_2)m_1 \\ 0 & s_2(m_2)m_1 \end{bmatrix} + \begin{bmatrix} \mu(m_2)s_2(m_1) & \mu(m_2)q_2(m_1) \\ 0 & 0 \end{bmatrix}
\end{aligned}$$

On comparing both sides we get $p_2(m_2)m_1 + \mu(m_2)q_2(m_1) = 0, \mu(m_2)s_2(m_1) = 0, s_2(m_2)m_1 = 0$. On following similar steps with $G_1 = \begin{bmatrix} 0 & 0 \\ n_1 & 0 \end{bmatrix}$ and $G_2 = \begin{bmatrix} 0 & 0 \\ n_2 & 0 \end{bmatrix}$, we get $r_3(n_2)n_1 = 0, s_3(n_2)n_1 + \nu(n_2)p_3(n_1) = 0, \nu(n_2)r_3(n_1) = 0$.

If Φ_{ad} has form $(\star)$ and satisfies condition (1) to (5), the assertion that Φ_{ad} is a anti σ-derivation on $\mathcal{G}$ will follow from direct computations. $\square$

Now we are ready to prove our main results:

Theorem 2.3. *Let $\mathcal{G} = \mathcal{G}(\mathcal{A}, \mathcal{M}, \mathcal{N}, \mathcal{B})$ be a 2-torsion free generalized matrix algebra over a commutative ring $\mathcal{R}$ with $\mathcal{A}$ and $\mathcal{B}$ have only trivial idempotents. An additive map $\Phi_J : \mathcal{G} \to \mathcal{G}$ is a Jordan σ-derivation on $\mathcal{G}$ if and only if Φ_J*

has the following form

$$\Phi_J\left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\right)$$
$$= \begin{bmatrix} p_1(a)+p_2(m)+p_3(n)-m_0\delta(b)n_0' & \gamma(a)m_0'+r_2(m)+r_3(n)+r_4(b) \\ s_1(a)+s_2(m)+s_3(n)-\delta(b)n_0' & n_0\gamma(a)m_0'+q_2(m)+q_3(n)+q_4(b) \end{bmatrix}, \tag{$\clubsuit$}$$

where $a \in \mathcal{A}; b \in \mathcal{B}; m, m_0, m_0{}' \in \mathcal{M}; n, n_0, n_0{}' \in \mathcal{N}$ *and* $p_1 : \mathcal{A} \to \mathcal{A}, r_2 : \mathcal{M} \to \mathcal{M}$, $s_3 : \mathcal{N} \to \mathcal{N}, q_4 : \mathcal{B} \to \mathcal{B}, p_3 : \mathcal{N} \to \mathcal{A}, r_4 : \mathcal{B} \to \mathcal{M}, s_1 : \mathcal{A} \to \mathcal{N}, q_2 : \mathcal{M} \to \mathcal{B}$ *are* $\mathcal{R}$*-linear maps satisfying the following conditions:*

1. $p_1(a^2) = p_1(a)a + \gamma(a)p_1(a) + \gamma(a)m_0s_1(a)$ *and* $p_1(mn) = -p_4(nm) + r_2(m)n + \mu(m)s_3(n)$;
2. $q_4(b^2) = q_4(b)b - \delta(b)n_0r_4(b) + \delta(b)q_4(b)$ *and* $q_4(nm) = -q_1(mn) + s_3(n)m + \nu(n)r_2(m)$;
3. $r_2(am) = p_1(a)m + \gamma(a)r_2(m) + \gamma(a)m_0q_2(m) + \mu(m)q_1(a)$,
$r_2(mb) = p_4(b)m + r_2(m)b - m_0\delta(b)q_2(m) + \mu(m)q_4(b)$;
4. $s_3(na) = s_3(n)a + q_1(a)n + n_0\gamma(a)p_3(n) + \nu(n)p_1(a)$,
$s_3(bn) = q_4(b)n - \delta(b)n_0p_3(n) + \nu(n)p_4(b) + \delta(b)s_3(n)$;
5. $r_3(bn) = r_3(n)b - m_0\delta(b)q_3(n), r_3(na) = \gamma(a)r_3(n), r_3(n)n = 0, \nu(n)r_3(n) = 0$;
6. $s_2(mb) = \delta(b)s_2(m), s_2(am) = s_2(m)a + n_0\gamma(a)p_2(m), \mu(m)s_2(m) = 0, s_2(m)m = 0$;
7. $p_2(m) = -m_0s_2(m) - \mu(m)n_0'$ *and* $q_3(n) = n_0r_3(n) + \nu(n)m_0'$;
8. $q_2(m) = n_0'm + n_0r_2(m)$ *and* $p_3(n) = m_0'n - m_0s_3(n)$;
9. $s_1(a) = -n_0'a - n_0\gamma(a)p_4(1) + n_0p_1(a)$ *and* $r_4(b) = -m_0'b - m_0q_4(b) + m_0\delta(b)q_1(1)$.

Proof. Suppose that Jordan σ-derivation has the following form as

$$\Phi_J\begin{bmatrix} a & m \\ n & b \end{bmatrix} = \begin{bmatrix} p_1(a)+p_2(m)+p_3(n)+p_4(b) & r_1(a)+r_2(m)+r_3(n)+r_4(b) \\ s_1(a)+s_2(m)+s_3(n)+s_4(b) & q_1(a)+q_2(m)+q_3(n)+q_4(b) \end{bmatrix},$$

where $p_1 : \mathcal{A} \to \mathcal{A}, p_2 : \mathcal{M} \to \mathcal{A}, p_3 : \mathcal{N} \to \mathcal{A}, p_4 : \mathcal{B} \to \mathcal{A}$; $r_1 : \mathcal{A} \to \mathcal{M}, r_2 : \mathcal{M} \to \mathcal{M}, r_3 : \mathcal{N} \to \mathcal{M}, r_4 : \mathcal{B} \to \mathcal{M}$; $s_1 : \mathcal{A} \to \mathcal{N}, s_2 : \mathcal{M} \to \mathcal{N}, s_3 : \mathcal{N} \to \mathcal{N}, s_4 : \mathcal{B} \to \mathcal{N}$ and $q_1 : \mathcal{A} \to \mathcal{B}, q_2 : \mathcal{M} \to \mathcal{B}, q_3 : \mathcal{N} \to \mathcal{B}, q_4 : \mathcal{B} \to \mathcal{B}$ are $\mathcal{R}$-linear maps. As Φ_J is the Jordan σ-derivation with automorphism σ

defined by $\Phi_J(G^2) = \Phi_J(G)G + \sigma(G)\Phi_J(G)$ for all $G \in \mathcal{G}$. Now we assume that $G = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$ and applying Lemma 2.1

$$\begin{aligned}
&\Phi_J\left(\begin{bmatrix} a^2 & 0 \\ 0 & 0 \end{bmatrix}\right) \\
&= \Phi_J\left(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\right)\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} + \sigma\left(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\right)\Phi_J\left(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\right) \\
&\begin{bmatrix} p_1(a^2) & r_1(a^2) \\ s_1(a^2) & q_1(a^2) \end{bmatrix} \\
&= \begin{bmatrix} p_1(a) & r_1(a) \\ s_1(a) & q_1(a) \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \gamma(a) & \gamma(a)m_0 \\ n_0\gamma(a) & 0 \end{bmatrix}\begin{bmatrix} p_1(a) & r_1(a) \\ s_1(a) & q_1(a) \end{bmatrix} \\
&= \begin{bmatrix} p_1(a)a & 0 \\ s_1(a)a & 0 \end{bmatrix} \\
&\quad + \begin{bmatrix} \gamma(a)p_1(a) + \gamma(a)m_0s_1(a) & \gamma(a)r_1(a) + \gamma(a)m_0q_1(a) \\ n_0\gamma(a)p_1(a) & n_0\gamma(a)r_1(a) \end{bmatrix}.
\end{aligned}$$

On comparing both sides we get $p_1(a^2) = p_1(a)a + \gamma(a)p_1(a) + \gamma(a)m_0s_1(a)$, $r_1(a^2) = \gamma(a)r_1(a)+\gamma(a)m_0q_1(a)$, $s_1(a^2) = s_1(a)a+n_0\gamma(a)p_1(a)$ and $q_1(a^2) = n_0\gamma(a)r_1(a)$.

Similarly, for $G = \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}$, we have $p_4(b^2) = -m_0\delta(b)s_4(b), r_4(b^2) = r_4(b)b - m_0\delta(b)q_4(b), s_4(b^2) = -\delta(b)n_0p_4(b) + \delta(b)s_4(b)$ and $q_4(b^2) = q_4(b)b - \delta(b)n_0r_4(b) + \delta(b)q_4(b)$.

Let us suppose $G = \begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}$, we get

$$\begin{aligned}
&\Phi_J\left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right) \\
&= \Phi_J\left(\begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}\right)\begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix} + \sigma\left(\begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}\right)\Phi_J\left(\begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}\right) \\
&\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} p_2(m) & r_2(m) \\ s_2(m) & q_2(m) \end{bmatrix}\begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & \mu(m) \\ 0 & 0 \end{bmatrix}\begin{bmatrix} p_2(m) & r_2(m) \\ s_2(m) & q_2(m) \end{bmatrix} \\
&= \begin{bmatrix} 0 & p_2(m)m \\ 0 & s_2(m)m \end{bmatrix} + \begin{bmatrix} \mu(m)s_2(m) & \mu(m)q_2(m) \\ 0 & 0 \end{bmatrix}
\end{aligned}$$

This leads to $0 = \mu(m)s_2(m), 0 = p_2(m)m + \mu(m)q_2(m)$ and $0 = s_2(m)m$. Similarly, choosing $G = \begin{bmatrix} 0 & 0 \\ n & 0 \end{bmatrix}$, we get $0 = r_3(n)n, 0 = q_3(n)n + \nu(n)p_3(n)$

and $0=\nu(n)r_3(n)$. Now choosing $G=\begin{bmatrix} a & m \\ 0 & 0 \end{bmatrix}$, we get

$$\begin{aligned}
p_2(am) &= p_2(m)a+\gamma(a)p_2(m)+\gamma(a)m_0s_2(m)+\mu(m)s_1(a),\\
r_2(am) &= p_1(a)m+\gamma(a)r_2(m)+\gamma(a)m_0q_2(m)+\mu(m)q_1(a),\\
s_2(am) &= s_2(m)a+n_0\gamma(a)p_2(m),\\
q_2(am) &= s_1(a)m+n_0\gamma(a)r_2(m).
\end{aligned}$$

Substitute $a=1$, we find that $p_2(m)=-m_0s_2(m)-\mu(m)s_1(1), 0=p_1(1)m+m_0q_2(m)+\mu(m)q_1(1), 0=n_0p_2(m)$ and $q_2(m)=s_1(1)m+n_0r_2(m)$. Also, if $G=\begin{bmatrix} 0 & m \\ 0 & b \end{bmatrix}$, then we get

$$\begin{aligned}
p_2(mb) &= -m_0\delta(b)s_2(m)+\mu(m)s_4(b),\\
r_2(mb) &= p_4(b)m+r_2(m)b-m_0\delta(b)q_2(m)+\mu(m)q_4(b),\\
s_2(mb) &= -\delta(b)n_0p_2(m)+\delta(b)s_2(m),\\
q_2(mb) &= s_4(b)m+q_2(m)b-\delta(b)n_0r_2(m)+\delta(b)q_2(m).
\end{aligned}$$

On putting $b=1$, we find that $p_2(m)=-m_0s_2(m)+\mu(m)s_4(1), 0=p_4(1)m-m_0q_2(m)+\mu(m)q_4(1), 0=n_0p_2(m)$ and $q_2(m)=-s_4(1)m+n_0r_2(m)$. Similarly, on assuming $G=\begin{bmatrix} a & 0 \\ n & 0 \end{bmatrix}$, we get

$$\begin{aligned}
p_3(na) &= p_3(n)a+r_1(a)n+\gamma(a)p_3(n)+\gamma(a)m_0s_3(n),\\
r_3(na) &= \gamma(a)r_3(n)+\gamma(a)m_0q_3(n),\\
s_3(na) &= s_3(n)a+q_1(a)n+n_0\gamma(a)p_3(n)+\nu(n)p_1(a),\\
q_3(na) &= n_0\gamma(a)r_3(n)+\nu(n)r_1(a).
\end{aligned}$$

Substitute $a=1$, we find that $p_3(n)=-r_1(1)n-m_0s_3(n), 0=m_0q_3(n), 0=q_1(1)n+n_0p_3(n)+\nu(n)p_1(1)$ and $q_3(n)=n_0r_3(n)+\nu(n)r_1(1)$. If $G=\begin{bmatrix} 0 & 0 \\ n & b \end{bmatrix}$, the we have

$$\begin{aligned}
p_3(bn) &= r_4(b)n-m_0\delta(b)s_3(n),\\
r_3(bn) &= r_3(n)b-m_0\delta(b)q_3(n),\\
s_3(bn) &= q_4(b)n-\delta(b)n_0p_3(n)+\nu(n)p_4(b)+\delta(b)s_3(n),\\
q_3(bn) &= q_3(n)b-\delta(b)n_0r_3(n)+\nu(n)r_4(b)+\delta(b)q_3(n).
\end{aligned}$$

If $b=1$, then $p_3(n)=r_4(1)n-m_0s_3(n), 0=-m_0q_3(n), 0=q_4(1)n-n_0p_3(n)+\nu(n)p_4(1)$ and $q_3(n)=n_0r_3(n)-\nu(n)r_4(1)$. Further we conclude that $s_4(b)=-\delta(b)n_0'$ and $r_1(a)=\gamma(a)m_0'$

Consider $G = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$, we get

$$\begin{aligned}
0 &= p_4(b)a + \gamma(a)p_4(b) + \gamma(a)m_0s_4(b) - m_0\delta(b)s_1(a), \\
0 &= r_1(a)b + \gamma(a)r_4(b) + \gamma(a)m_0q_4(b) - m_0\delta(b)q_1(a), \\
0 &= s_4(b)a + n_0\gamma(a)p_4(b) - \delta(b)n_0p_1(a) + \delta(b)s_1(a), \\
0 &= q_1(a)b + n_0\gamma(a)r_4(b) - \delta(b)n_0r_1(a) + \delta(b)q_1(a).
\end{aligned}$$

On putting $b = 1$, we find that

$$\begin{aligned}
0 &= p_4(1)a + \gamma(a)p_4(1) + \gamma(a)m_0s_4(1) - m_0s_1(a), \\
0 &= r_1(a) + \gamma(a)r_4(1) + \gamma(a)m_0q_4(1) - m_0q_1(a), \\
0 &= s_4(1)a + n_0\gamma(a)p_4(1) - n_0p_1(a) + s_1(a), \\
0 &= q_1(a) + n_0\gamma(a)r_4(1) - n_0r_1(a) + q_1(a).
\end{aligned}$$

Substitute $a = 1$, we find that

$$\begin{aligned}
0 &= p_4(b) + p_4(b) + m_0s_4(b) - m_0\delta(b)s_1(1), \\
0 &= r_1(1)b + r_4(b) + m_0q_4(b) - m_0\delta(b)q_1(1), \\
0 &= s_4(b) + n_0p_4(b) - \delta(b)n_0p_1(1) + \delta(b)s_1(1), \\
0 &= q_1(1)b + n_0r_4(b) - \delta(b)n_0r_1(1) + \delta(b)q_1(1).
\end{aligned}$$

This implies that $2q_1(a) = -2n_0\gamma(a)m_0'$ and $2p_4(b) = 2m_0\delta(b)n_0'$. Now choosing $G = \begin{bmatrix} 0 & m \\ n & 0 \end{bmatrix}$, we get

$$\begin{aligned}
&\Phi_J\left(\begin{bmatrix} mn & 0 \\ 0 & nm \end{bmatrix}\right) \\
&= \Phi_J\left(\begin{bmatrix} 0 & m \\ n & 0 \end{bmatrix}\right)\begin{bmatrix} 0 & m \\ n & 0 \end{bmatrix} + \sigma\left(\begin{bmatrix} 0 & m \\ n & 0 \end{bmatrix}\right)\Phi_J\left(\begin{bmatrix} 0 & m \\ n & 0 \end{bmatrix}\right) \\
&\begin{bmatrix} p_1(mn) + p_4(nm) & r_1(mn) + r_4(nm) \\ s_1(mn) + s_4(nm) & q_1(mn) + q_4(nm) \end{bmatrix} \\
&= \begin{bmatrix} p_2(m) + p_3(n) & r_2(m) + r_3(n) \\ s_2(m) + s_3(n) & q_2(m) + q_3(n) \end{bmatrix}\begin{bmatrix} 0 & m \\ n & 0 \end{bmatrix} \\
&\quad + \begin{bmatrix} 0 & \mu(m) \\ \nu(n) & 0 \end{bmatrix}\begin{bmatrix} p_2(m) + p_3(n) & r_2(m) + r_3(n) \\ s_2(m) + s_3(n) & q_2(m) + q_3(n) \end{bmatrix}.
\end{aligned}$$

This leads to $p_1(mn) + p_4(nm) = r_2(m)n + \mu(m)s_3(n), r_1(mn) + r_4(nm) = p_3(n)m + \mu(m)q_3(n), s_1(mn) + s_4(nm) = q_2(m)n + \nu(n)p_2(m)$ and $q_1(mn) + q_4(nm) = s_3(n)m + \nu(n)r_2(m)$.

Converse is trivial.

□

Theorem 2.4. *Let $\mathcal{G} = \mathcal{G}(\mathcal{A}, \mathcal{M}, \mathcal{N}, \mathcal{B})$ be a 2-torsion free generalized matrix algebra over a commutative ring $\mathcal{R}$ with $\mathcal{A}$ and $\mathcal{B}$ have only trivial idempotents. If bilinear pairings $\xi_{\mathcal{MN}} = 0 = \Omega_{\mathcal{NM}}$, then every Jordan σ-derivation can be written as the sum of a σ-derivation and anti-σ-derivation.*

Proof. From the Theorem 2.3, we have

$$\begin{aligned}
&\Phi_J\left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\right) \\
&= \begin{bmatrix} p_1(a)+p_2(m)+p_3(n)+m_0\delta(b)n_0' & \gamma(a)m_0'+r_2(m)+r_3(n)+r_4(b) \\ s_1(a)+s_2(m)+s_3(n)-\delta(b)n_0' & -n_0\gamma(a)m_0'+q_2(m)+q_3(n)+q_4(b) \end{bmatrix} \\
&= \begin{bmatrix} p_1(a)+p_2(m)+p_3(n)+m_0\delta(b)n_0' & \gamma(a)m_0'+r_2(m)+r_4(b) \\ s_1(a)+s_3(n)-\delta(b)n_0' & -n_0\gamma(a)m_0'+q_2(m)+q_3(n)+q_4(b) \end{bmatrix} \\
&\quad + \begin{bmatrix} 0 & r_3(n) \\ s_2(m) & 0 \end{bmatrix} \\
&= \Phi_d\left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\right) + \Phi_{ad}\left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\right).
\end{aligned}$$

This shows that Jordan σ-derivation can be expressed the sum of a derivation σ-derivation Φ_d and an anti σ-derivation Φ_{ad}, which is the desired result. □

Corollary 2.5. *Let $\mathcal{T}_n(\mathcal{R})(n \geq 2)$ be the upper (or lower) triangular matrix algebra over 2-torsion free commutative ring $\mathcal{R}$ with identity. Then every Jordan σ-derivation on $\mathcal{T}_n(\mathcal{R})(n \geq 2)$ is a σ-derivation.*

Proof. It can be easily seen that $\mathcal{T}_n(\mathcal{R})(n \geq 2)$ is a generalized matrix algebra in which bilinear pairings are zero. From Theorem 2.4, every Jordan σ-derivation on $\mathcal{T}_n(\mathcal{R})(n \geq 2)$ can be written as the sum of a σ- derivation and an anti-σ-derivation. In view of [11, Corollary 2.5] we conclude that the part of anti σ-derivation is zero. This leads to the fact that every Jordan σ-derivation on $\mathcal{T}_n(\mathcal{R})(n \geq 2)$ is a σ-derivation. □

3 Topics for future research

The main aim of this paper is to concentrate on studying σ-derivations on generalized matrix algebras. The current work together with [8, 16] indicate that it is feasible to investigate Lie σ-derivations on generalized matrix algebras by moderate adaption of current methods. We have good reasons to believe that characterizing Lie σ-derivations on generalized matrix algebras is also of great interest. In the light of the motivation and contents of this article, we would like to end this article by proposing several potential questions.

Let $\mathcal{A}$ be a unital algebra over a commutative ring $\mathcal{R}$, σ be the automorphisms on $\mathcal{A}$. A map $L : \mathcal{A} \to \mathcal{A}$ is called a Lie σ-derivation if

$$L([x,y]) = L(x)y + \sigma(x)L(y) - L(y)x - \sigma(y)L(x)$$

for all $x, y \in \mathcal{A}$. Obviously, if $\sigma = I$, then Lie σ-derivation is called Lie derivation.

Recently, many authors studied Lie derivation on various kind of algebras [6, 7, 10]. The first characterization of Lie derivations was obtained by Martindale [12] in 1964 who proved that every Lie derivation on primitive ring can be written as a sum of derivation and an additive mapping of ring to its center that maps commutators into zero, i.e, Lie derivation has the standard form. Cheung [3] established the structures of commuting maps and Lie derivation on triangular algebras. Further, Yang and Zhu [16] characterized the additive σ-derivation on triangular algebras. Li and Wei [8] studied the structure of Lie derivations on generalized matrix algebras and prove that it has standard form. Now here it is natural to raise a question:

Question 3.1. *Let L be Lie σ-derivation on generalized matrix algebra $\mathcal{G}$. How can we describe its general form? That is* $L\begin{bmatrix} a & m \\ n & b \end{bmatrix} = \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}$ *for every* $\begin{bmatrix} a & m \\ n & b \end{bmatrix} \in \mathcal{G}$.

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